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Two-parametric zeta function regularization in superstring theory

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Abstract

In this paper some quite simple examples of applications of the zeta-function regularization to superstring theories are presented. It is shown that the Virasoro anomaly in the BRST formulation of (super)strings can be directly computed from the original expressions of the operators as well as normal ordering constants and masses of ground levels. Hawking's zeta regularization is recognized as an efficient tool for direct calculations, bringing no ambiguities.

Possible implications for global GSO operators' phases definitions (maybe ensuring modular invariance) will be discussed elsewhere.

This paper is dedicated to my friend Lenka.

arch-ive/9510105

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1 Simple examples, usual Riemann zeta function

I begin with the simplest determination of the bosonic string critical dimension. In the light-cone gauge the operator p^-

$$p^- = \frac{1}{2\pi p^+}\int_0^\pi d\sigma \cdot (\pi^2 p(\sigma)^2 + x'^2) \tag{1}$$

has form (after translating to modes α_n)

$$p^- = \frac{1}{2p^+}(p^{i2} + m^2) \tag{2}$$

where we used (now we consider open strings)

$$p^{i2} + m^2 = \sum_{n\in Z} \alpha^i_n \alpha^i_{-n}. \tag{3}$$

Here α_n are annihilation operators for $n > 0$, so if we want to write the expression in a normal-ordered form, we must change the order of the operators for $n > 0$ terms, according to the commutation relation

$$[\alpha^i_m, \alpha^j_n] = m^1 \delta_{m+n}\delta_{ij}. \tag{4}$$

Then we get ($\alpha^i_0 = p^i$)

$$m^2 = 2\sum_{n=1}^{\infty}(\alpha^i_{-n}\alpha^i_n + (d-2)\frac{n^1}{2}), \tag{5}$$

where $(d-2) = \delta^{ii}$ means number of transversal coordinates. The first part gives a finite contribution when operating on a state, particularly the first term annihilates the ground level (tachyon). The second part is a clearly divergent sum which must be regularized (see next section). Analytic continuation according to the exponent 1 in n^1 gives

$$\sum_{n=1}^{\infty} n^1 = 1+2+3+4+\ldots = 0+1+2+3+\ldots = \zeta(-1) = -\frac{1}{12}. \tag{6}$$

I hope that the reader will not be disturbed too much by the equations between finite numbers and divergent sums. (To be provocative, they really equal.)

The first excited level of open string $\alpha^i_{-1}|0\rangle$ has only $(d-2)$ degeneracy, so it can't form a massive representation of the subgroup $SO(d-1)$ of the

full Lorentz group, fixing some $d-$momentum vector of this level. It means that the first level (for which $\sum \alpha^i_{-n}\alpha^i_n = 1$) must be massless.

$$m^2 = 2\left(\sum_{n=1}^{\infty}(\alpha^i_{-n}\alpha^i_n) - \frac{d-2}{24}\right) \tag{7}$$

Condition $m^2 = 0$ gives $d = 26$.

1.1 A quick analytic continuation

Let's consider a bit generalized Riemann zeta function with a parameter s

$$\zeta_s(x) = \sum_{n=1}^{\infty}(n+s)^{(-x)}, \tag{8}$$

where the most usual value of s will be $s = 0$. This formula is convergent for $Re\, s > 1$, for instance $\zeta(2) = \pi^2/6$ (exactly). Let's note that if we add 1 to s, the only difference will be that the first term drops ($n = 1$). We can use Taylor series according to the parameter s.

$$\zeta_{s+1}(x) = \zeta_s(x) - (1+s)^{-x} = \zeta_s(x) + \frac{1}{1!}\frac{\partial\zeta_{s'}(x)}{\partial s'}|_{s'=s} + \frac{1}{2!}\frac{\partial^2\zeta_{s'}(x)}{\partial s'}|_{s'=s} + \dots \tag{9}$$

But the derivatives with respect to s are computed easily:

$$\frac{\partial}{\partial s}\zeta_s(x) = \sum_{n=1}^{\infty}(n+s)^{-x} = (-x)\zeta_s(x+1) \tag{10}$$

and m-th derivative

$$\frac{\partial^m}{\partial s^m} = (-x)(-x-1)\dots(-x-m+1)\zeta_s(x+m) \tag{11}$$

So we have

$$\begin{aligned} -(1+s)^{-x} = (-x)\zeta_s(x+1) + \frac{(-x)(-x-1)}{2!}\zeta_s(x+2) + \\ + \frac{(-x)(-x-1)(-x-2)}{3!}\zeta_s(x+3) + \dots \end{aligned} \tag{12}$$

Let's substitute $x \to 0$. Since for $x > 1$ zeta function is finite and the terms are multiplied by $(-x) \to 0$, on the right hand side just the first term survives, e.g.

$$\lim_{x\to 0} x\zeta_s(x+1) = 1 \tag{13}$$

$\zeta_s(y)$ has pole for $y \to 1$. Substituing $x \to -1$ gives

$$-1 + s = \zeta_s(0) + \frac{1}{2!}(-x-1)\zeta_s(x+2)$$

But $\lim_{x\to -1}(-x-1)\zeta_s(x+2) = -1$, therefore

$$\zeta_s(0) = -\frac{1}{2} - s \tag{14}$$

Substituing $x \to -2$ gives equation

$$-(1+s)^2 = 2\zeta_0(-1) + \frac{1}{2!}2\cdot 1\zeta_s(0) + \frac{1}{3!}2\cdot 1\cdot(-x-2)\zeta_0(x+3)$$

which after short algebra

$$\begin{aligned} -(1+s)^2 &= 2\zeta_s(-1) + (-\frac{1}{2} - s) + (-\frac{1}{3}), \\ \zeta_s(-1) &= -\frac{1}{12} - \frac{s+s^2}{2} = \frac{1}{24} - \frac{(s+1/2)^2}{2} \end{aligned} \tag{15}$$

Without details we mention also ($x \to -3$)

$$\zeta_s(-2) = -\frac{s(s+1/2)(s+1)}{3}. \tag{16}$$

Other values we give for $s = 0$ only

$$\zeta_0(-1) = -\frac{1}{12},\ \zeta_0(-3) = \frac{1}{120},\ \zeta_0(-5) = -\frac{1}{252} \tag{17}$$

Interesting fact that zeta of negative even number vanishes

$$\zeta_0(-2) = \zeta_0(-4) = \zeta_0(-6) = \ldots = 0 \tag{18}$$

can be proved by mathematical induction if we sum the Taylor series around $s = 0$ for $s = +1$ and $s = -1$ but I will not enter to details.

1.2 Consistency of the regularization

Presented formulas for the divergent sums have many characteristics of consistency. For example, if we make s increase by one, the result decreases by the first term (this fact was used in the derivation).

For example (for other cases similarly)

$$1+\sum_{n=y+1}^{\infty} n^0 = 1+\frac{1}{2}-(y+1) = \sum_{n=y}^{\infty} n^0. \tag{19}$$

So we regularize in fact really only the part "in far infinity" and with any finite number of terms we can manage as with normal numbers.

Next interesting property states that

$$\sum_{n\in Z} (n+s)^x = 0 \quad \text{for } x = 0,1,2,\dots. \tag{20}$$

This identity is quite important in checking the independence of the commutators on a particular form of the commutants. Particularly, $\sum_{n=1}^{\infty} n^0 = \sum_{n=-1}^{-\infty} n^0 = -1/2$ and these $-1/2$'s together with 1 arising from $n=0$ give zero.

2 Modifications of formulas for the regularization

In the first example we noted that the regularization parameter is the exponent over the mode's index. It means that in a sense we compute the result of a regularized parameter for a general (complex) degree of the derivatives in this expression, then we realize that the result is an analytic function of these parameters that can be continued to the interesting values for the degrees.

Particularly, we must modify some formula for (anti)commutators and so on. So for instance, instead of $\{c_m, b_n\} = \delta_{m+n}$ we write a more precise equation

$$\{c_m, b_n\} = \delta_{m+n}\cdot n^0, \quad \{c_m, c_n\} = \{b_m, b_n\} = 0. \tag{21}$$

Here we must understand why there is no contradiction in computing sum $\sum_{n=1}^{\infty} n^0 = 1+1+1+\ldots = -1/2$. Someone could find it counterintuitive since if we add 1 to this sum, we get some $1+1+1+\ldots$ sum again. But the right hand side should equal $+1/2$. In fact, there is no contradiction here since we must always remember from which mode the number 1 arises. In other words, we should always write it as n^0.

3 Virasoro anomaly

Different contributions to the (super)Virasoro algebra's anomaly can be evaluated by worldsheet methods, but we can also use calculations involving

modes. So for example, in the page I/130 of [GSW], authors argue that the easiest and safest way to determine the anomaly is by evaluating specific matrix elements. But here I wish present a proper way for its direct calculation.

3.1 Ghost contribution

In this subsection L_m will always denote $L_m^{(gh)}$. Parameter J is the conformal dimension of the antighost and $J = 2$ for the ordinary antighost b. We define the ghost contributions to Virasoro algebra generators as

$$L_m = \sum_{n \in Z} (m(J-1) - n)\, b_{m+n} c_{-n}. \tag{22}$$

This expression equals its normal-ordered for $m \neq 0$. For $m = 0$ it differs from its normal-ordered part by a c-number, which we again compute by zeta-function regularization. (Let's remark that in [GSW] they always mean $: L_m :$ when they use symbol L_m.) Products of ghost operators must be exchanged for $n > 0$ (or $n \geq 0$ which gives the same result) in the following sum and a normal ordering constant appears.

$$L_0 = \sum_{n \in Z} (-n) b_n c_{-n} =: L_0 : + \sum_{n=1}^{\infty} (-n) \cdot n^0 =: L_0 : + \frac{1}{12} \tag{23}$$

Our main task is to compute the commutators of two L_m's.

$$\begin{aligned}[L_m, L_{m'}] = \sum_{n,n' \in Z} & (m(J-1) - n)(m'(J-1) - n') \cdot \\ & \cdot (b_{m+n}\{c_{-n}, b_{m'+n'}\}c_{-n'} - b_{m'+n'}\{c_{-n'}, b_{m+n}\}c_{-n})\end{aligned} \tag{24}$$

The bottom line contains two terms such that the second can be obtained from the first one by $(m, n \leftrightarrow m', n')$. We use the anticommutators from the previous section. Kronecker's delta will reduce the summation over n, n' to only one summation.

$$[L_m, L_{m'}] = \ldots = \sum_{n \in Z} (m(J-1) - n)(m'J - n) b_{m+n} c_{m'-n} \cdot n^0 - (m, m' \leftrightarrow n, n')$$

The ghost oscillators c, b in the last expression can be exchanged for $m+m' \neq 0$ since their anticommutator equals zero. Then the result contains only a finite number of terms with creation operators $c_{k<0}, b_{k<0}$ as the last factors

in the products. Therefore n^0 can be replaced by 1 and terms can be summed classically (no divergent sum need to be regularized) and we get

$$[L_m, L_{m'}] = \ldots = (m - m')L_{m+m'} + A(m)\delta_{m+m'}. \tag{25}$$

For $m+m' = 0$ we must compute also the anomaly term $A(m)$. If $m' = -m$

$$[L_m, L_{-m}] = \sum_{n\in Z}(m(J-1)-n)(-mJ-n)b_{m+n}c_{-m-n}\cdot n^0 - (m \leftrightarrow -m) = \tag{26}$$

Using substitution $n = N - m$ e.g. $N = n + m$ we get

$$= \sum_{N\in Z}(mJ-N)(m(1-J)-N)b_N c_{-N}(N-m)^0 - (m \leftrightarrow -m) =$$

The terms with $N > 0$ should be rewritten in a normal-ordered fashion. The q-number part combines with $N \leq 0$ terms giving $2m:L_0:$ and the c-number part appears

$$= 2m:L_0: + \left(\sum_{N=1}^{\infty} m^2 J(1-J)N^0(N-m)^0 - \right.$$
$$\left. - \sum_{N=1}^{\infty} mN^1(N-m)^0 + \sum_{N=1}^{\infty} N^2(N-m)^0 - (m \leftrightarrow -m)\right) \tag{27}$$

We involve a new variable $J = (1+k)/2$ and use results of double-parametric zeta regularization.

$$= 2m:L_0: + \left(\frac{m^2}{4}(1-k^2)(-\frac{1}{2}+\frac{m}{2}) + \frac{m}{12} - \frac{m^3}{4} + \frac{m^3}{6} - (m \leftrightarrow -m)\right)$$

The final result reads

$$[L_m, L_{-m}] = 2m:L_0: + \frac{1}{12}(1-3k^2)m^3 + \frac{m}{6} = 2mL_0 + \frac{1}{12}(1-3k^2)m^3. \tag{28}$$

We can notice that for L_m we get a simpler expression containing m^3 term only than if we use $:L_0:$ (without the natural normal ordering constant). If we translate L_m to $T_{++}(\sigma)$, then only the $\delta'''(\sigma-\sigma')$ anomaly appears.

3.2 Bosonic coordinates' contribution

In this subsection L_m will denote $L_m^{(x)}$. Let's accept

$$L_m = \frac{1}{2} \sum_{n \in Z} \alpha_{m-n} \alpha_n, \quad [\alpha_m, \alpha_n] = m^1 \delta_{m+n}. \tag{29}$$

Then L_0 contains again a natural normal-ordering constant.

$$L_0 =: L_0 : + \frac{1}{2} \sum_{n=1}^{\infty} n^1 =: L_0 : - \frac{1}{24}. \tag{30}$$

Since there are 26 coordinates (26 equal contributions to this constant), the total normal-ordering constant is (using last subsection)

$$26 L_0^{(x)} + L_0^{(gh)} =: 26 L_0^{(x)} + L_0^{(gh)} : - \frac{26}{24} + \frac{1}{12} =: 26 L_0^{(x)} + L_0^{(gh)} : -1. \tag{31}$$

Evaluation of the commutator looks like

$$\begin{aligned}[L_m, L_{m'}] = \frac{1}{4} \sum_{n,n' \in Z} (&\alpha_{m-n} [\alpha_n, \alpha_{m'-n'}] \alpha_{n'} + [\alpha_{m-n}, \alpha_{m'-n'}] \alpha_n \alpha_{n'} + \\ &+ \alpha_{m'-n'} \alpha_{m-n} [\alpha_n, \alpha_{n'}] + \alpha_{m'-n'} [\alpha_{m-n} \alpha_{n'}] \alpha_n) = \end{aligned} \tag{32}$$

For $m + m \neq 0$ the result again equals its normal-ordered part, so factors like $(n' - m')^1$ can be replaced by $(n' - m')$ and summed together. So the result has general form

$$[L_m, L_{m'}] = (m - m') L_{m+m'} + A(m) \delta_{m+m'}. \tag{33}$$

The anomaly can be computed using

$$\sum_{N=1}^{\infty} (N + m)^1 N^1 = \zeta_{0,m}(-1, -1) = \frac{m^3 - m}{12} \tag{34}$$

Finite anomaly equals

$$[L_m, L_{m'}] = 2m : L_0 : + \frac{m^3 - m}{12} = 2m L_0 + \frac{m^3}{12} \tag{35}$$

The cancellation with ghost contribution now gives the critical dimension in form $(k = 3)$; $d = 3k^2 - 1 = 26$.

Anomalies in other commutators and anticommutators can be computed in similar fashion.

3.3 Two-parametric zeta function

During the calculations we often needed not only sums like $\sum_n n^x$ but also a more general

$$\zeta_{s,t}(x,y) = \sum_{n=1}^{\infty} (n+s)^{-x}(n+s+t)^{-y} \tag{36}$$

This reduces to previous zeta functions for $t = 0$

$$\zeta_{s,0}(x,y) = \zeta_s(x+y) \tag{37}$$

Also, the sum should be independent on ordering of the product inside the sum, e.g.

$$\zeta_{s+t,-t}(y,x) = \zeta_{s,t}(x,y) \tag{38}$$

The sums for general (nonzero) t can be effectively computed using Taylor expansion according to t.

Direct substitutions gives for $x + y \to 0$

$$\zeta_{s,t}(1+x,y) = \frac{1}{x+y} + \text{finite part.} \tag{39}$$

Calculation of e.g. $\zeta_{s,t}(-m,0)$ runs as follows (we write the same ε to both parameters which turns to be the simplest way to obey equation (38) – more precise ways bring the same result)

$$\zeta_{s,t}(-m+\varepsilon,\varepsilon) = \zeta_{s,0}(-m,0) + (-\varepsilon)t\zeta_{s,0}(-m,1) + \dots \tag{40}$$

Here only the first term $\propto t^0$ of the Taylor sum and the term proportional to $\zeta(1+x,y)$, $x+y \to 0$ contribute, giving

$$\zeta_{s,t}(-m,0) = \zeta_s(-m) - \frac{t^{m+1}(-1)^m}{2(m+1)}. \tag{41}$$

Other values can be deduced in a similar manner.

Here I summarize some useful formulas.

$$\sum_{n=y}^{\infty} n^0 = \frac{1}{2} - y, \quad \sum_{n=y}^{\infty} n^1 = -\frac{1}{12} + \frac{y-y^2}{2}, \quad \sum_{n=y}^{\infty} n^2 = -\frac{1}{3}y(y-\frac{1}{2})(y-1)$$

$$\sum_{n=y}^{\infty} n^0(n+m)^0 = \frac{1}{2} - y - \frac{m}{2}, \quad \sum_{n=y}^{\infty} n^1(n+m)^0 = -\frac{1}{12} + \frac{y-y^2}{2} + \frac{m^2}{4}$$

$$\sum_{n=y}^{\infty}(n+m)^1 n^0 = -\frac{1}{12}+\frac{y-y^2}{2}+\frac{m}{2}(1-2y-\frac{m}{2}) \tag{42}$$
$$\sum_{n=y}^{\infty} n^1(n+m)^1 = -\frac{1}{3}y(y-1)(y-\frac{1}{2}+\frac{3}{2}m)+\frac{m^3-m}{12}$$
$$\sum_{n=y}^{\infty}(n-\frac{m}{2})^1(n+\frac{m}{2})^1 = -\frac{1}{3}y(y-1)(y-\frac{1}{2})+\frac{m^2}{4}(y-\frac{1}{2})$$

4 Anticommutator of worldsheet SUSY currents in 4F models

In this section I show rather surprising fact in 4F models (FFFF models means models in the four-dimensional free fermionic formulation) that the anticommutator of the worldsheet SUSY current with itself gives a correct result, the energy-momentum tensor containg derivatives of the fermion fields. The zeta-function regularization is being used. The origin of the derivatives in the result is again similar to the emergency of anomalies and normal-ordering shifts of ground levels.

We repeat the supercurrent in the case of six compactified dimensions from [af].

$$T_F(z) = \psi^\mu \partial_z X_\mu + \sum_{i=1}^{6} \chi^i y^i \omega^i, \tag{43}$$

where $\mu = 0,1,2,3$, ψ_μ are the worldsheet superpartners of bosonic coordinates X_μ and χ, y, ω are also hermitean fermionic fields.

Now the anticommutators of two T should give an observable proportional to energy-momentum. ($J_+ = T_F$.)

$$\{J_+(\sigma), J_+(\sigma')\} \propto \delta(\sigma-\sigma')T_{++}(\sigma). \tag{44}$$

Anticommutator of the first four terms $\psi^\mu \partial_z X_\mu$ gives an expected result. But it seems hard at first look to obtain terms like $i \cdot y^i \partial_\sigma y^i$ (and also for the fields ω, χ) from anticommutator of terms $\chi y \omega$ containing no derivatives, which have anticommutation relations as

$$\{\chi(\sigma),\chi(\sigma')\} = \{y(\sigma),y(\sigma')\} = \{\omega(\sigma),\omega(\sigma')\} \propto \delta(\sigma-\sigma'). \tag{45}$$

But these derivative terms are obtained due to the similar phenomenon which causes also the normal-ordering constants, anomalies in (super)Virasoro algebra, term proportional to c_0 in the BRST operator and other…

The calculation will be only sketched here and overall normalization will be ignored. Writing $y(\sigma)$ in terms of modes $\sum_{n\in Z} e^{2in\sigma} y_n$ (and identical sums for χ and ω), and also $J_+ = \sum_n F_n e^{2in\sigma}$ we get

$$\{F_m, F_{m'}\} \propto \sum_{\substack{n_1+n_2+n_3=m \\ n'_1+n'_2+n'_3=m'}} y_{n_2}\omega_{n_3} y_{n'_2}\omega_{n'_3}\delta_{n_1+n'_1}(n_1)^0 + (\chi, y, \omega \text{ cycl.perm.}) \tag{46}$$

Cyclic permutations can be rewritten in a similar way as the first term.

$$\sum_{n_2,n'_2,n_3} y_{n_2} y_{n'_2}\omega_{n_3}\omega_{m+m'-n_2-n'_2-n_3}(m-n_2-n_3)^0 \tag{47}$$

Such an expression can be expressed in the normal-ordered form. No product of four operators can survive since for such terms the zeroth power can be omitted and ($n_2 \leftrightarrow n'_2$) exchange ensures the cancellation.

Only terms with two operators remain. Exchange of the two y's can contribute by a c-number which keeps only $\omega\omega$ terms and vice versa. (Also a total c-number anomaly in the total anticommutator survives but I will not enter to details here.)

Let us have a look to the $\omega\omega$ terms. They arise from $y_{n_2} y_{n'_2}$ for $n_2 + n'_2 = 0$, $n'_2 \leq 0$. So the result looks something like

$$\sum_{n_2=0'}^{\infty} \sum_{n_3} (n_2)^0 \omega_{n_3}\omega_{m+m'-n_3}(m-n_2-n_3)^0. \tag{48}$$

Symbol $0'$ expresses that only half of the $n_2 = 0$ term is summed. Since $\sum_{n_2=0'}^{\infty}(n_2)^0(n_2+n_3-m)^0 = 1/2 - n_3/2 + m/2 - 1/2 = m/2 - n_3/2$, we obtain

$$\frac{1}{2}\sum_{n_3}(m-n_3)\omega_{n_3}\omega_{m+m'-n_3}. \tag{49}$$

Because of anticommuting the term $\propto m$ vanishes (or gives only c-number) and the result can be written also as a mode of $i\omega\partial_\sigma\omega$

$$\propto \sum_{n_3} n_3\omega_{n_3}\omega_{m+m'-n_3}. \tag{50}$$

4.1 Other constants in 4F models

Here I mention that the equation (2) of [af]

$$M_L^2 = -\frac{1}{2} + \frac{\alpha_L \cdot \alpha_L}{8} + N_L = -1 + \frac{\alpha_R \cdot \alpha_R}{8} + N_R = M_R^2 \tag{51}$$

has a natural explanation after using formula

$$\sum_{n=y}^{\infty} n^1 = \frac{1}{24} - \frac{1}{2}(y - \frac{1}{2})^2. \tag{52}$$

Also expression (4) of [af] for the $U(1)$ generator

$$Q(f) = \frac{1}{2}\alpha(f) + F(f) \tag{53}$$

can be deduced from the continual definition $\propto \int_0^\pi d\sigma f f^*$, which after prescription to modes gives

$$\sum_{n \in Z + \frac{\alpha(f)+1}{2}} f_{-n} f_n^* = \dots \tag{54}$$

(since frequencies of creation operators f_{-n} are in $Z + (\alpha(f)+1)/2$, e.g. $\alpha = 0$ is antiperiodic boundary condition and $\alpha = 1$ periodic), which can be translated to normal-ordered form by the usual changing order in terms $n < 0$

$$\dots = \sum_{n \in Z + \frac{(1-\alpha(f))}{2}} f_n f_{-n}^* =: \mathrm{that} : + \frac{1}{2}\alpha(f) = F(f) + \frac{1}{2}\alpha(f). \tag{55}$$

4.2 Global GSO operators' phases

It seems possible that even GSO operators may be defined globally for all the sectors. Different choices of signs and phases of GSO operators in different sectors correspond to different forms how can be these operators written. (They create only finite numbers of non-equivalent theories.) For example, by multiplication the operator by

$$\exp(2\pi i \sum f_n f_{-n}^*) \tag{56}$$

we change its sign only in periodic sector ($\sum_{n=1}^{\infty} n^0 = -1/2$) while in antiperiodic sector it remains constant ($\sum_{n=1/2}^{\infty} n^0 = 0$).

Models with many sectors can be constructed by defining a group of unitary GSO operators Ξ. Physical state is an eigenstate of all these operators corresponding to eigenvalue 1. (This condition remains also for nontrivial sectors.) So in a sense we "set the operators equal one" – therefore we must involve for each GSO operator $G \in \Xi$ a sector where a identical operation

(particularly rigid shift of a closed string by π) has the same effect as the operator itself. (G_{NS} is the operator anticommuting with all the fermionic fields, to ensure that the trivial sector of group Ξ is the *antiperiodic* one.)

$$G' \cdot L \cdot G'^{-1} = L_{\text{after } \sigma \to \sigma+\pi}, \quad G' = G \cdot G_{NS} \tag{57}$$

One of the simplest examples of this process is compactification on circle. Here the group of sectors (group of GSO operators Ξ) is isomorphic to Z. GSO operators are given by ($n \in Z$) rigid shift of string by multiple of vector a

$$\exp(ina \cdot p_{\text{zero}}). \tag{58}$$

We must include sectors with non-zero winding numbers (where the shift $\sigma \to \sigma + \pi$ moves the string by na).

More conventional example of GSO operator is that change phases of some of fermionic fields. (Here g_b means a global phase correction of the operator G_b given by some vector b.)

$$G_b = g_b \exp \sum_f b(f) \kappa \int f f^* d\sigma \tag{59}$$

where κ is a conventional constant such that $\kappa \int f f^* d\sigma = \sum f_n f^*_{-n}$.

5 Summary

Zeta function regularization (given by analytical continuation of the expressions according to the degrees of derivatives) was shown as a reliable method giving correct results in many cases and author argued for using similar operations with divergent sums as the most direct way to do the calculations.

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